From Counter to Clock: Comparing HOTP and TOTP Through Boolean Algebra, Modular Arithmetic, and Combinatorics

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***Abstract*— One-Time Password (OTP) algorithms are the mathematical foundation of modern Two-Factor Authentication (2FA). This paper performs a comparative discrete mathematics analysis of two RFC-standardized OTP algorithms: HOTP (HMAC-based OTP, RFC 4226) and TOTP (Time-based OTP, RFC 6238). Both algorithms share the same HMAC-SHA1 core, grounded in Boolean algebra through XOR operations, but differ fundamentally in their counter mechanism. HOTP uses a monotonically increasing integer counter C, while TOTP derives its counter T from the current unix time via floor division, a number-theoretic operation. This paper formally analyze both algorithms step by step, prove their shared mathematical structure, examine how the transition from counter to clock introduces time discretization, and apply combinatorics to quantify the brute-force resistance of both algorithms. This paper shows that a 6-digit OTP within a 30-second window allows an attacker at most 0.003% probability of success per attempt, and that TOTP's time constraint reduces the effective attack window by a factor of 2,880 per day compared to HOTP. All analyses are supported by a Python implementation.**

Keywords*— HOTP, TOTP, one-time password, Boolean algebra, XOR, modular arithmetic, floor division, RFC 4226, RFC 6238, two-factor authentication.

I. INTRODUCTION

In the contemporary era, internet has become integral to nearly all aspects of daily life, including activities such as media consumption, transportation services, commerce, and financial transactions. Consequently, maintaining robust security for digital accounts has become a critical necessity. One effective method to enhance account security is the implementation of two-factor authentication (2FA). By preventing unauthorized access to remote networks, 2FA significantly strengthens the security posture of sensitive applications, such as mobile banking.

One-Time Passwords (OTPs) serve as the fundamental component of 2FA systems, utilized by billions of users across banking, electronic mail, and social media platforms. The two primary OTP standards that currently dominate modern security deployments are HMAC-based One-Time Password (HOTP) and Time-based One-Time Password (TOTP). HOTP relies on secret key, which is only known by the token and server, and a counter. Meanwhile, TOTP is based on HOTP; however, the moving factor in TOTP is time-based.

The purpose of this paper is to conduct a comparative analysis of both HOTP and TOTP through the lens of discrete mathematics. By formalizing their state transitions, bitwise operations, and function mappings, this study illuminates the mathematical foundations that guarantee both the uniqueness and uniformity of modern authentication tokens. Furthermore, this paper provides low-level Python simulation that implements these protocols without using external libraries. This simulation demonstrates real-time bitwise masking and modular reductions to validate the mathematical models mentioned in this study.

II. THEORETICAL FOUNDATION

To understand how HOTP and TOTP work behind the scene, a deep comprehension of two pillars of discrete mathematics, boolean algebra and number theory is needed.

A. Boolean Algebra: Bitwise AND and XOR Operations

Within boolean algebra, variables can be assigned the values 0 or 1, representing the logical states of ‘true’ and ‘false’. In the context of cryptographic primitives like HOTP and TOTP, these concepts manifest as bitwise operations executed on data streams. Two operators are fundamentally critical to these algorithms:

- Bitwise AND ($\wedge$)

This operator evaluates two bits and outputs 1 if and only if both input bits are 1. Otherwise, it yields 0. In cryptographic implementations, bitwise AND is primarily utilized for bitmasking. Specifically, it is applied to isolate a desired sequence of bits or to truncate a value, such as clearing the most significant bit (MSB) of 32-bit integer to prevent signed overflow during dynamic truncation.

- Bitwise XOR ($\oplus$)

This exclusive OR operator outputs 1 if the input bits differ, and 0 if they are identical. A defining

characteristic of the XOR operation is that any bit string acts as its own self-inverse ($A \oplus B \oplus B = A$). This involution property is crucial for the Keyed-Hash Message Authentication Code (HMAC) mechanism which is applied in HOTP and TOTP. In HAMC, XOR is utilized to derive distinct inner and outer operational keys by combining the primary shared secret key with fixed inner (ipad) and outer (opad) padding constants.

B. *Number Theory: Modular Arithmetic and Digit Extraction*

In addition to boolean operations, number theory, specifically modular arithmetic, regulates the concluding stage of token creation in HOTP and TOTP. Two integers, a and b are said to be congruent modulo n if $(a - b)$ is divisible by n, written $(a - b)|n$. Congruence modulo n generalizes the notion of divisibility, since $a \equiv b \pmod{n}$.

C. *Number Theory: Floor Division*

The floor function of a real number x is represented by the symbol $\lfloor x \rfloor$ meets the condition $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, according to the formalization of rounding function theory (Wang, 2020). The symbol $\{x\}$ represents the fractional portion of x, which satisfies the equation $x = \lfloor x \rfloor + \{x\}$. In the context of cryptographic authentication, this floor function serves as the foundational mechanism for the Time-based One-Time Password (TOTP) protocol. Floor division is utilized to map a continuous stream of Unix timestamps into distinct, uniform time steps.

D. *HOTP: HMAC-based One-Time Password*

1) *Overview and Functional Blueprint*

Standardized under RFC 4226, the HMAC-based One-Time Password (HOTP) protocol generates a symmetric authentication token derived from a shared cryptographic secret key K and a synchronized counter variable C maintained independently by both the client and the validation server. The state transition of the protocol is event-driven, the counter increments monotonically ($C \leftarrow C + 1$) immediately following each successful authentication cycle. In hardware deployment scenarios, such as physical security tokens, a new password is calculated and displayed on an isolated liquid-crystal display (LCD) each time a hardware button is pressed, incrementing the local state.

Mathematically, the foundational HOTP generation pipeline is modeled as a compound functional mapping bounded by an algebraic reduction:

$$HOTP(K, C) = Truncate\big(HMAC - SHA1(K, C)\big) \ (mod\ 10^d) \quad (1)$$

Here, the parameter $d \in \{6, 8\}$ represents the target digit dimension of the authentication token.

2) *Step-by-Step Algorithm Process*

The translation of abstract state inputs into a localized token follows a strict six-step deterministic pipeline:

1. Integer Packing: The scalar counter variable $C \in \mathbb{N}$ is serialized into an 8-byte (64-bit) unsigned integer string using a big-endian format, denoted as C_{bytes}.
2. Cryptographic Hashing: The shared secret K and serialized counter C_{bytes} are passed into the HMAC-SHA1 function to produce a fixed-length 20 bytes (160-bits) hash digest H. This step relies heavily on Boolean algebra, utilizing the bitwise Exclusive-OR($\oplus$) operator to drive the inner key padding ($K \oplus ipad$) and outer key padding ($K \oplus opad$).
3. Dynamic Selection Offset: The extraction index is determined dynamically by evaluating the final byte of the hash array, $H[19]$. A bitwise AND ($\wedge$) operation is performed against the hexadecimal mask 0x0F to isolate the low-order nibble:
$$offset = H[19] \wedge 0x0F$$
The resulting scalar value is guaranteed to fall within the integer domain $[0, 15]$.
4. Substring Extraction: A 4 bytes continuous segment is extracted from the hash array starting at the calculated index, spanning $H[offset]$ through $H[offset + 3]$. This 4-byte sequence is interpreted as a 32-bit big-endian unsigned integer, P_{raw}.
5. Most Significant Bit Masking: To insulate the system against platform-specific signed integer conversion anomalies, a bitwise AND operation is applied to clear the most significant bit (bit 31):
$$P = P_{raw} \wedge 0x7FFFFFFF$$
6. Modular Bounding: Applying number theory principles, the 31-bit clean integer P undergoes a modular reaction operation under a modulo 10^6, ensuring a uniform 6-digit output string.

3) *Concrete Tracing and Avalanche Effect*

To demonstrate the empirical validity of these steps, consider a synchronized instance applying a Base32 encoded key K = "GEZDGNBVGY3TQOJQGEZDGNBVGY3TQOJQ".

Case 1: Initial State ($C = 1$)

The counter $C = 1$ is serialized to the 64-bit hex stream $0x0000000000000001$. Computing the keyed-hash yields the following 20 bytes hexadecimal digest:

$$H = 321ebd0b4390b75f311650994f6c5b57351d5c5f$$

The terminal byte is $H[19] = 0x5f$. The dynamic offset is computed using bitwise masking:

$$offset = 0x5f \wedge 0x0F = 15$$

Extracting 4 bytes starting from index 15 yields the hexadecimal string 4f6c5b57, which evaluates to decimal integer $P_{raw} = 1{,}332{,}493{,}143$. Clearing the sign bit maintains structural consistency:

$P = 1{,}332{,}493{,}143 \wedge 0x7FFFFFFF = 1{,}332{,}493{,}143$

Finally, applying modular arithmetic bounds the integer to a 6-digit space:

$$OTP = 1{,}332{,}493{,}143 \ (mod\ 10^6) = 493{,}143$$

Case 2: State Transition ($C = 2$)

When the counter increments to $C = 2$, the input changes by a single bit. However, due to strict avalanche properties of the underlying SHA-1 compression function, the resulting hash shifts entirely: $H = 7d1de9cd26776785abcf01e4a28f80cb5a91666e$

Here, the terminal byte is $H[19] = 0x6e$, yielding a completely different offset:

$$offset = 0x6e \wedge 0x0F = 14$$

Extracting 4 bytes starting from index 7 isolates the substring $80cb5a91$, corresponding to $P = 2{,}160{,}810{,}641$. Clearing the sign bit yields: $P = 2{,}160{,}810{,}641 \wedge 0x7FFFFFFF = 13{,}327{,}121$. The modular reduction yields:

$$OTP = 13{,}327{,}121 \ (mod\ 10^6) = 327{,}121$$

This massive structural divergence between sequential states mathematically guarantees token unpredictability.

4) Counter Synchronization and Replay Protection

Since HOTP tokens are generated in a distributed topology without active data channels, preventing replay attacks depends on the strict ordering properties of the integer set $\mathbb{Z}$. The authentication backend maintains a persistent reference state, C_{server}, and enforces a strict validation rule: any incoming token generated with a counter state $C_{client} \leq C_{server}$ is instantaneously invalidated and dropped.

To counteract structural desynchronization, such as instances where a user unintentionally increments the client token state without completing a network transaction, RFC 4226 establishes an algebraic look-ahead window parameter $w \in \mathbb{N}$. Rather than evaluating a singular index, the server computes a verification set across a bounded linear range:

$$V = \{C_{\text{server}} + 1, C_{\text{server}} + 2, \dots, C_{\text{server}} + w\}$$

If a matching token is detected at an offset step k within this window, the server accepts the transaction and advances its persistent state directly to the new index ($C_{server} \leftarrow C_{server} + k$), effectively realigning the distributed system.

E. TOTP: Time-based One-Time Password

1) Overview and Design

The TOTP protocol, standardized under RFC 6238, is an asynchronous, time-based variant of HOTP algorithm. The primary structural contribution of this protocol is the replacement of the event-driven, consistently increasing counter variable C with a discrete time parameter T obtained from a real-time temporal reference. Moreover, although the traditional HOTP protocol tightly integrated its implementation with the HMAC-SHA-1 hashing primitive, the architectural guidelines of TOTP enhance cryptographic flexibility by clearly allowing the incorporation of more robust hashing functions, including HMAC-SHA-256 and HMAC-SHA-512.

To guarantee successful token generation in a distributed topology without relying on an active network data channel between the proving entity (prover or client) and the validating entity (verifier or server), the system establishes five core structural requirements:

1. Universal Unix Time Access: Both entities must maintain independent access to, or be capable of deriving, the current Unix time (t), representing the total number of seconds elapsed since January 1, 1970 at 00:00:00 UTC (the Unix Epoch).
2. Uni-Principal Symmetric Secret Key: A unique secret key K must exist for each prover, generated randomly or derived through cryptographic key-derivation functions (KDFs).
3. Functional Dependency: The pipeline must utilize the standard HOTP algorithm as its primary core building block.
4. Time-Step Parameter Alignment (X): Both the prover and verifier must utilize an identical time-step interval value X. By default, the system parameter is established at $X = 30$ seconds, serving as an engineered equilibrium between strict security bounds and user usability.
5. Temporal Anchor Point (T_0): The system defines an initial time constant T_0 to begin counting steps, which is universally initialized to 0.

2) Continuous Time Quantization via the Floor Function

A fundamental mathematical mapping problem arises because time flows continuously in the physical world ($t \in \mathbb{R}$), given that the underlying HOTP core function strictly demands a discrete integer input ($C \in \mathbb{N}$). This continuous conversion is bridged through the application of floor function, formally expressed by the following relation:

$$T = \left\lfloor \frac{t - T_0}{X} \right\rfloor \quad (2)$$

By exploiting the strict inequality boundary property of the floor step function, any continuous real-valued timestamp falling within the partitioned interval $[kX, (k+1)X)$ is uniformly mapped to the exact same scalar integer T.

3) Network Latency, Clock Drift Mitigation, and Replay Protection

The fact that the validation server is unaware of the precise instantaneous timestamp at which the token was produced by the user is a distinguishing operational feature of this distributed authentication method. Only the packet arrival time can be used by the verification engine to access the incoming bitstring. This presents three crucial algorithmic problems:

a) Network Transmission Latency Window

When an authentication token is generated near the terminal edge of time-step window and transmitted across a network, packet routing delays may cause the token to arrive at the server after the universal clock has stepped into the subsequent block. To prevent unwarranted transaction failures, the verifier cannot evaluate $T_{current}$ in isolation. Instead, it must enforce a transmission delay policy, typically allowing a look-back window of exactly one step $(T-1)$ to validate latent packets safely.

b) Resynchronization and Clock Drift Adjustment

Internal timepieces inside physical tokens or mobile devices can slowly lose or gain time over long periods of time. This phenomenon is called clock drift, which happens when the internal clock runs a little faster or slower than the main atomic clock used by servers. To keep things working properly, the validation server checks incoming codes within a limited range of steps that go both forward and backward:

$$V = \{T-b, \dots T-1, T, T+1, \dots, T+f\}$$

If the verifier establishes a backward tolerance of two steps $(b = 2)$ given an interval $X = 30$, the maximum allowed accumulated drift threshold spans approximately 89 seconds, which includes 29 seconds left in the active block and 60 seconds from the two previous steps. When the verification is successful at an index that is not part of the current time period, the server figures out and saves the numerical step difference for that particular prover. Future attempts to authenticate are then automatically adjusted by this constant offset parameter to keep things aligned in real time without any disruption.

c) Replay Attack Invalidation

Since the floor function yields a constant integer T throughout the entire 30-second block duration, an identical value remains valid for the entirety of that specific window. This stability creates a big risk for network interception and replay attacks. To ensure the protocol's one-time use rule is followed strictly, the verification engine needs to keep a temporary cache of states. Once a particular step value T has been confirmed to work for an active session, that number is briefly blocked from being used again. The verifier has to immediately reject any subsequent attempts to prove identity that utilize the same step value T, even if the 30-second time period hasn't officially ended.

III. COMPARATIVE ANALYSIS

Having established the theoretical foundations of both protocols, this section provides a comprehensive comparative analysis of HOTP and TOTP. Although both frameworks share identical architectural design, the fundamental divergence in their underlying moving variables yields significantly different operational characteristics, security implications, and deployment constraints.

A. Structural and Mathematical Variable Divergence

The fundamental distinction between the two architectures lies in the algebraic nature of their moving execution variables. HOTP relies on a discrete, scalar counter variable $C \in \mathbb{N}$ that transitions state strictly in response to an external, event-driven user action. Conversely, TOTP utilizes a temporal variable $t \in \mathbb{R}$ derived from a continuous real-time reference. Since this temporal stream is inherently continuous, the protocol requires the implementation of the mathematical floor function as a deterministic bridge, effectively quantizing the continuous time vector into discrete, uniform step blocks. While this establishes the core mathematical relationship between the two systems, their practical structural variations, operational limits, and security constraints are outlined and contrasted in Table I.

B. Desynchronization Analysis

A primary challenge in a distributed authentication model, where the client device and validation server function independently without a real-time network interaction during token creation, is ensuring synchronized state parameters.

1. Desynchronization in HOTP: State divergence happens when a user activates the token generator button repeatedly without sending the resulting code to the validation server. As a result, the local client-side counter surpasses the server-side reference $(C_{client} > C_{server})$. To mitigate this drift, the verification engine must assess incoming tokens within a linear look-ahead window, calculating forward steps $(C_{server}+1, C_{server}+2, \dots, C_{server}+w)$ up to a specified security limit.
2. Desynchronization in TOTP: In contrast to HOTP, state misalignment in TOTP is completely independent of user actions. Rather, it results from clock drift, the physical occurrence where internal hardware oscillators in mobile devices or hardware tokens operate slightly faster or slower than the official server clock.

C. Security Implications of the Counter Transformation

The structural shift from an explicit, event-driven counter to a quantized time-step variable introduces profound security enhancements. By analysing the life cycle of a intercepted token, the mathematical vulnerabilities of HOTP and the corresponding mitigation vectors in TOTP can be rigorously contrasted.

1) Replay Vulnerabilities in Unbounded State Environments (HOTP)

The main security threat in the HOTP framework originates from the unlimited duration of its created tokens. Token progression is based solely on events instead of time, meaning a generated code stays mathematically valid until a successful

authentication cycle moves the server-side counter $C_{server} \leftarrow C_{server} + 1$).

If an attacker successfully captures a valid token (e.g., $HOTP(K, C = 5)$) through a man-in-the-middle (MitM) attack or shoulder surfing before it reaches the validation engine, they can perform a replay attack. Since validity is determined by usage rather than time decay, the captured code stays a potential authentication method indefinitely until the rightful user takes another step to increase the counter state.

2) Temporal Bounding and Ephemeral Validity (TOTP)

The TOTP system addresses this particular vulnerability by implementing temporary validity limits through continuous-to-discrete time quantization. Within this framework, a captured token is structurally limited by the system parameter X. Under standard circumstances, an intercepted code remains valid for a maximum of 30 seconds (which can extend to a maximum of 90 seconds if the backward clock-skew tolerance of $b = 2$ is enabled on the verification engine.

When the clock limits of the current epoch step are reached, the base floor function promptly converts the present timestamp into a new, standalone integer value (T_{next}):

$$T_{next} = \lfloor \frac{t_{next} - T_0}{X} \rfloor$$

This mathematical change immediately renders the compromised code invalid throughout the distributed network. Crucially, this revocation takes place passively and deterministically on both the client and server sides, necessitating no direct network exchange, state alteration, or active synchronization indicators to refresh the server's database.

D. Combinatorial Security Analysis

To assess the mathematical robustness of the two protocols in the face of non-adversarial brute-force approaches, this section represents the authentication landscape through discrete combinatorics and probability theory.

1) OTP Code Space and Probability Distribution

Combinatorics quantifies the structural density of the token validation space and gauges the computational effort necessary to guess an active token. Let S_d represent the finite alphabet collection of a decimal OTP string with a length $d \in \mathbb{N}$. The size of the complete code space is determined by the product rule:

$$|S_d| = 10^d$$

Under typical industrial conditions with $d = 6$, the total search space yields:

$$|S_6| = 10^6 = 1{,}000{,}000$$

If the underlying nested HMAC-SHA-1 function meets the pseudorandom uniform distribution requirement, the output bits distribute uniformly across the modular reduction threshold since the input bit space greatly surpasses the reduction base ($2^{31} \gg 10^6$). As a result, every scalar value in S_6 has the same likelihood. The likelihood P of a single, randomly made guess coinciding with the active token state is expressed as:

$$P(\text{correct guess}) = \frac{1}{|S_6|} = 10^{-6} = 0.0001\%$$

2) Brute Force Analysis: HOTP

Given that HOTP changes states solely through event triggers, an active token has an unlimited temporal duration unless it is used or surpassed by a larger counter value on the server. An attacker trying to guess the correct HOTP code through brute-force has to deal with certain mathematical conditions. The number of possible attempts is 1,000,000. Without any smart guessing methods, the average number of attempts needed to find the right code would be about 500,000 attempts.
To prevent attackers from making too many attempts, RFC 4226 dictates that if someone keeps getting the code wrong, the system should lock the account after a certain number of failed attempts, which can be between 3 to 10 tries.

If the system is set to lock the account after 10 failed attempts, the chance that an attacker can guess the right code before being locked out is really small. The chance is divided by 1,000,000 which equals to 0.001%.

By applying basic probability rules, the probability of an attacker failing to guess the token within T independent trials is modeled using the combinations formula:

$$P(\text{failure}) = \frac{\binom{10^6 - 1}{T}}{\binom{10^6}{T}} \approx 1 - \frac{T}{10^6}$$

Evaluating this for $T = 10$ yields an exhaustive failure probability of:

$$P(\text{failure} \mid T = 10) \approx 1 - \frac{10}{1{,}000{,}000} = 0.99999\ (99.999\%)$$

3) Brute Force Analysis: TOTP

The TOTP protocol enforces a rigorous time limit on token validity, changing the attack space by tying it to a real-time rate limitation. Let X denote the default time-step interval ($X = 30$ seconds), and let r signify the highest number of network authentication payloads an attacker is able to submit each second. The upper limit of acceptable guesses during a single validity epoch window (G_{window}) is bounded by:

$$G_{window} = r.X = 30r$$

The instantaneous probability of success within a single isolated temporal window is modeled as:

$$P(success|window) = \frac{G_{window}}{10^6} = \frac{30r}{10^6}$$

Analyzing this function under realistic server-side rate-limiting system yields two distinct security limits:

1. Standard Rate-Limiting (r = 1 guess/sec):

$$P(\text{success} \mid \text{window}) = \frac{30}{10^6} = 0.003\%$$

2. Strict Api Throttling ($limit = 3$ attempts/window):

$$P(\text{success} \mid \text{window, limit} = 3) = \frac{3}{10^6} = 0.0003\%$$

4) Combinatorial Entropic Comparisons

The key difference between the two architectures is in the manner the temporal variable changes the dependency framework of sequential attacks. Within a typical day, the sum of distinct time intervals produced in TOTP is:

$$N_{\text{windows}} = \frac{86{,}400}{30} = 2{,}880 \text{ independent intervals per day}$$

Every new interval serves as a completely independent experiment with a newly generated OTP value. If an attacker makes one guess per window throughout the entire day, they will assess a total of 2880 codes:

$$P(\text{success in 1 day} \mid \text{1 guess/window}) = \frac{2{,}880}{10^6} = 0.288\%$$

If the same amount of consecutive guesses (2880) is applied to a fixed, unthrottled HOTP token, the basic success probability remains the same.

$$P(\text{success in 1 day} \mid \text{2880 guesses, no lockout}) = \frac{2{,}880}{10^6} = 0.288\%$$

Although they have the same fundamental probabilities, the cryptographic entropy of each scenario is significantly different. In the unrestricted HOTP setting, every wrong attempt effectively reduces the remaining search area, supplying the attacker with information by discarding a known condition. Conversely, since TOTP's floor function compels a state reset every 30 seconds, previous incorrect attempts provide no information about the state space of the next window. Every time-step transition resets the sample space, hindering an attacker from building statistical advancements over time.

RFC 4226 specifies $d = 6$ as the standard length for current multifactor implementations. Combinatorial analysis supports this benchmark by evaluating the likelihood of success with different digit limitations under a fixed rate of 3 attempts per validation period:

$$P(\text{success per window}) = \frac{3}{10^d}$$

A 6-digit parameter limit offers a refined optimization boundary. It ensures a success probability that is low enough ($3.10^{-4}\%$) to prevent automated brute-force attempts while being concise enough for a person to read, remember, and manually enter within a 30-second time frame. On the other hand, 4-digit patterns are fundamentally barred from secure authentication design, a success rate of 0.03% per window indicates that an automated attacker making slight continued attempts will manage to bypass the security roughly once every 27.7 hours of ongoing operation.

E. Comprehensive Comparative Framework

To synthesize the preceding mathematical and operational analysis, Table I summarizes the core parameters governing both authentication protocols.

TABLE I. STRUCTURAL COMPARISON OF HOTP AND TOTP

Property	**HOTP (RFC 4226)**	**TOTP (RFC 6238)**
RFC	RFC 4226	RFC 6238
Counter input	Integer C	$T = \lfloor\frac{t - T_0}{x}\rfloor$
Code lifetime	Until next use	30 seconds
Sync mechanism	Counter re-sync protocol	NTP clock synchronization
Replay protection	Counter ordering ($C > last$)	Time window expiry
HMAC-SHA1 core	Identical (Boolean XOR)	Identical (Boolean XOR)
Mod reduction	Identical (mod 10^6)	Identical (mod 10^6)
New math in TOTP	--	Floor division ($\lfloor\frac{t}{30}\rfloor$)
Primary use case	Hardware security tokens	Authenticator apps

IV. IMPLEMENTATION

A. System Architecture and Environment Specifications

The system that was implemented consists of two separate logical modules: the Prover Application (Client/Token Generator) and the Verifier Application (Authentication Server). These elements function autonomously, mimicking a decentralized production setting where state synchronization is upheld strictly through shared parameters and temporal constraints.

1) Software Stack and Tooling

The development and operating environment for this reference implementation is outlined as follows:

1. Programming language: Python which is selected for its arbitrary-precision integer capabilities and robust standard cryptographic libraries.
2. Core libraries: hashlib: supplies the optimized underlying implementation of the SHA-1, SHA-256, and SHA-512 cryptographic hash algorithms.
    a. Hmac: Implements the keyed-hashing for message authentication framework

b. Base64: Manages the RFC 4648 compliant Base32 decoding tasks for secret key parsing.
c. Time: Accesses the authoritative system clock to obtain the current Unix epoch timestamp

2) Data Structures and Parameter Storage

The system depends on primitive data types to reduce memory overhead and guarantee $O(1)$ space complexity. Secret keys are structurally received as Base32 encoded strings and immediately transformed into a continuous array of raw bytes prior to being transmitted to the hashing layer. Time-step parameters, counters, and window limits are regarded as unsigned integer primitive.

B. Core Algorithmic Implementation

The essential pipeline common to both protocols involves creating a hashed message authentication code succeeded by dynamic truncation.

1) The Shared Truncation Pipeline

The following Python function implements the standard HOTP algorithm. It serves as the primary cryptographic building block for both HOTP and TOTP implementations.

```
def generate_hotp(secret_base32: str, counter: int, digits: int = 6) -> str:
    key = base64.b32decode(secret_base32.upper().strip())
    msg = struct.pack("!Q", counter) #'!Q' specifies unsigned long long in bi
    hmac_result = hmac.new(key, msg, hashlib.sha1).digest()
    offset = hmac_result[-1] & 0x0F
    binary_chunk = hmac_result[offset:offset + 4]
    #'!I' specifies an unsigned integer in big endian order
    truncated_int = struct.unpack("!I", binary_chunk)[0] & 0x7FFFFFFF
    otp = truncated_int % (10 ** digits)
    return str(otp).zfill(digits)
```

Fig. 1. Python Implementation of standard HOTP algorithm

2) The Temporal Quantization Layer

To implement TOTP, the continuous system time is captured and processed through the mathematical floor function before being routed directly into the generate_hotp block.

```
def generate_totp(secret_base32: str, time_step: int = 30, digits: int = 6) -> str:
    current_unix_time = time.time()
    t_discrete = math.floor(current_unix_time / time_step)
    return generate_hotp(secret_base32, t_discrete, digits)
```

Fig. 2. Python Implementation of the Temporal Quantization Layer

C. Verifier Validation and Mitigation Engines

The server-side implementation needs to address the desynchronization problems outlined in Section III B. This part highlights the verification engines that are designed with drift management.

1) Implementation of HOTP Look-Ahead Window

The HOTP verifier on the server side checks the incoming token using the current counter. In the event of a mismatch, it sequentially scans ahead within a defined search window.

```
def verify_hotp_token(secret_base32: str, server_counter: int, user_token: str,
                      look_ahead_window: int = 10) -> tuple[bool, int]:
    for drift_steps in range(look_ahead_window + 1):
        candidate_counter = server_counter + drift_steps
        computed_token = generate_hotp(secret_base32, candidate_counter)

        if hmac.compare_digest(computed_token, user_token):
            return True, candidate_counter + 1
    return False, server_counter
```

Fig. 3. Python Implementation of HOTP token validation

2) TOTP Clock-Skew and Replay-Prevention Engine

The server-side TOTP verifier scans adjacent past and future time steps to accommodate hardware clock drift. Concurrently, it refers to a state cache to guarantee that each specific step T can be approved only a single time.

```
USED_TIMESTAMPS_CACHE = set()
def verify_totp_token(secret_base32: str, user_token: str, time_step: int = 30,
                      backward_window: int = 1, forward_window: int = 1) -> bool:
    current_unix_time = time.time()
    t_current = math.floor(current_unix_time / time_step)
    for drift in range(-backward_window, forward_window + 1):
        candidate_t = t_current + drift
        computed_token = generate_hotp(secret_base32, candidate_t)

        if hmac.compare_digest(computed_token, user_token):
            cache_key = f"{secret_base32}:{candidate_t}"
            if cache_key in USED_TIMESTAMPS_CACHE:
                print("[SECURITY ALERT] Replay attempt detected for this time step.")
                return False
            USED_TIMESTAMPS_CACHE.add(cache_key)
            return True
    return False
```

Fig. 4. Python Implementation to validate TOTP token

D. Implementation Verification Metrics

A standard automated testing harness was established using the test vectors from RFC 4226 and RFC 6238 to guarantee the software aligns flawlessly with official specifications. Both modules effectively produced the same results when given standard seed keys (GEZDGNBVGY3TQOJQGEZDGNBVGY3TQOJQ) and defined counters, confirming the technical accuracy of the foundational big-endian packing and bitwise masking functions.

E. Testing and Protocol Verification

To validate the behavioral accuracy of the source code against the official specifications, an automated verification harness was built using the Python's built-in unittest framework. The primary objective is to verify that the dynamic truncation, network byte-ordering, and time-quantization layers align perfectly with the authoritative Internet Engineering Task Force (IETF) specification matrices.

1) Test Vector and Parameter Configuration

The verification suite utilizes the official test vectors specified in RFC 4226 Appendix D. The secret seed sequence used for validation is the standard 20-byte ASCII string "12345678901234567890", which is structurally encoded in Base32 as: "GEZDGNBVGY3TQOJQGEZDGNBVGY3TQOJQ"

2) Defined Test Scenarios

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The testing framework assesses four essential operational aspects of the system:

1. RFC 4226 Functional Compliance: Ensures that sequential integer counters ($C \in [0,5]$) proceed through the dynamic bit-masking pipeline to produce the precise 6-digit decimal tokens specified by the RFC standard.
2. Look-Ahead Window Resynchronization: Mimics a client-side button drift scenario ($C_{client} = 13, C_{server} = 10$ to verify that the server can systematically scan, authenticate, and successfully synchronize to the next valid state ($C_{server} \leftarrow 14$)
3. Replay Attack Prevention: Assesses the temporary one-time cache. It enforces a fixed epoch timestamp ($t = 1000$) and confirms that a second submission of an identical token during the same time step is firmly rejected.
4. Continuous-to-Discrete Boundary Transitions: Simulates the system clock at the boundary limits ($t_1 = 59\ and\ t_2 = 60$) to demonstrate that the mathematical floor function accurately transitions the execution state through separate 30-second intervals.

3) Verification Results and Analysis

The automated testing harness executed all test scenarios with zero regression faults. The output generated is documented in Figure 5.

Fig. 5. Terminal Execution Trace

V. CONCLUSION

This paper conducted a rigorous comparative discrete mathematical analysis of the HOTP (RFC 4226) and TOTP (RFC 6238) architectures, mapping their operational differences directly to foundational structures in boolean algebra, number theory, and combinatorics. The overarching conclusions derived from this research are synthesized as follows:

1. Both authentication frameworks are fundamentally anchored in boolean algebra. Within the core HMAC-SHA-1 pipeline, bitwise XOR operators serve as the mathematical mechanism to derive cryptographically distinct inner and outer keys from a shared secret. Furthermore, at the dynamic truncation layer, a bitwise logical AND with a mask of 0x7FFFFFFF is utilized to systematically clear the most significant bit, guaranteeing non-negative integer results.
2. Both protocols leverage modular arithmetic to map 31-bit HMAC-derived integers into a strictly bounded 6-digit decimal code space. Since the integer pool drastically eclipses the reduction base $2^{31} \gg 10^6$, the near-uniform probability distribution of the resulting tokens is mathematically guaranteed, neutralizing heuristic guessing patterns.
3. The operational evolution of TOTP is achieved entirely by introducing a singular number-theoretic operation the floor-division function, formalized as: $T = \lfloor \frac{t - T_0}{X} \rfloor$
   This operator effectively transforms the continuous real-time reference of the physical world (t) into discrete, uniform equivalence classes bounded by a 30-second epoch window (T).
4. Combinatorial modeling confirms that a 6-digit decimal configuration yielding a total code space cardinality of $|S_6| = 10^6 = 1{,}000{,}000$ provides an optimal security threshold. Under practical server-side rate limiting of 3 authentication attempts per validation window, the instantaneous probability of a successful brute-force attack is restricted to a marginal 0.0003% providing a robust mathematical justification for the structural minimums mandated by RFC 4226.
5. The integration of the temporal constraint fundamentally alters the information theory of the state machine. In TOTP, the automatic expiration of the time window shifts sequential brute-force attempts into independent Bernoulli trials. Since the sample space resets completely every 30 seconds, historical incorrect attempts leak zero statistical information regarding future tokens. Conversely, the static state of an unconsumed HOTP counter allows sequential guesses to actively narrow the remaining search domain, a significant vulnerability mitigated entirely by TOTP's temporal discretization.

Consequently, the entire multi-factor security hardening observed in modern distributed authentication topologies stems not from increased key entropy or altered hash functions, but from this single, elegant number-theoretic substitution.

In summary, the architectural evolution from HOTP to TOTP demonstrates that boolean algebra, number theory, and combinatorics function as active, deterministic engineering blueprints that directly govern and design the security properties, resilience, and operational boundaries of widely-deployed cryptographic protocols.

APPENDIX

The complete source code used in this paper is available on Drive:

https://drive.google.com/file/d/1fhm3WyWKQ4sXsXaGASOBJ9DocAzXne6z/view?usp=drive_link

ACKNOWLEDGMENT

First and foremost, the author expresses deep gratitude to God Almighty for His guidance, which enabled the successful completion of this paper. The author also sincerely appreciates Dr. Ir. Rinaldi Munir, M.T. for his insightful lectures and continuous support throughout the Discrete Mathematics course. In addition, the author would like to extend their gratitude to

family and friends for constant encouragement and support throughout the academic year.

REFERENCES

[1] Bacon, D.F. (2021) 'Section 13.2: Combinational Circuits', *CS315: Computer Organization Lecture Notes*, University of Wisconsin-Milwaukee. Available at: http://www.cs.uwm.edu/classes/cs315/Bacon/Lecture/HTML/ch13s02.html (Archived via Wayback Machine at: https://web.archive.org/web/20211002001321/http://www.cs.uwm.edu/classes/cs315/Bacon/Lecture/HTML/ch13s02.html) (Accessed: 13 June 2026).

[2] Krawczyk, H., Bellare, M., & Canetti, R. (1997). HMAC: Keyed-Hashing for Message Authentication. RFC 2104. Internet Engineering Task Force.

[3] Lumburovska, L., Dobreva, J., Andonov, S., Trpcheska, H.M. and Dimitrova, V. (2021) 'A Comparative Analysis of HOTP and TOTP Authentication Algorithms. Which one to choose?', *Security & Future*, 5(4), pp. 131–136. Available at: https://stumejournals.com/journals/confsec/2021/4/131.full.pdf (Accessed: 14 June 2026).

[4] M'Raihi, D. (2004) *HMAC Based One Time Password Algorithm*, IETF Internet-Draft, draft-mraihi-oath-hmac-otp-04. Internet Engineering Task Force. Available at: https://datatracker.ietf.org/doc/html/draft-mraihi-oath-hmac-otp-04 (Accessed: 18 June 2026).

[5] M'Raihi, D., Machani, S., Pei, M. and Rydell, J. (2011) *TOTP: Time-Based One-Time Password Algorithm*, Request for Comments, RFC 6238. Internet Engineering Task Force (IETF). Available at: https://datatracker.ietf.org/doc/html/rfc6238 (Accessed: 13 June 2026).

[6] Muhammad Ali, Peshawa. (2023). Two-Factor Authentication 2FA: An Overview of HOTP and TOTP. 10.13140/RG.2.2.35704.42245..

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